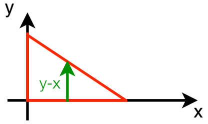
Flächenintegrale

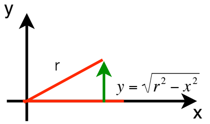
Dreiecksfläche

$$V = \int_0^x \int_0^{y-x} f(x,y)\, dy\, dx$$



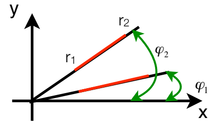
Kreisfläche (kartesisch)

$$V = \int_0^x \int_0^{\sqrt{r^2-x^2}} f(x,y)\, dy\, dx$$



Kreisfläche (polar)

$$V = \int_{\varphi_1}^{\varphi_2} \int_{r_1}^{r_2} f(x,y)\cdot r\, dr\, d\varphi$$



$$\begin{aligned} \textit{kartesisch:}\quad & x\cdot y \\ \textit{polar:}\quad & r\cdot\cos(\varphi)+r\cdot\sin(\varphi) \\ & = \frac{1}{2} r^2 \sin(2\varphi) \end{aligned}$$

$$r^2 = x^2 + y^2$$

Rotationssymmetrische Körper polar berechnen. Ist in vielen Fällen einfacher.

Mehrfachintegrale berechnen

$$\int_{x_1}^{x_2} \int_{y_1}^{y_2} f(x,y)\, dy\, dx = \int_{x_1}^{x_2} \left[\int_{y_1}^{y_2} f(x,y)\, dy \right] dx$$

1. f nach y integrieren und Grenzen einsetzen
2. f nach x integrieren und Grenzen einsetzen

Bsp.

$$V = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 1\, dy\, dx$$

$$= \int_{-1}^{1} [y]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\, dx$$

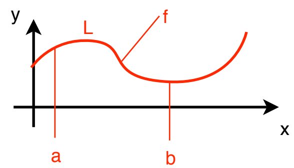
$$= \int_{-1}^{1} 2\cdot\sqrt{1-x^2}\, dx$$

$$= \left[x\sqrt{1-x^2} + \arcsin(x)\right]_{-1}^{1} = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$$

Kurvenlänge

Länge L

$$L = \int_a^b \sqrt{1 + f'(x_i)^2}\, dx$$



Oberfläche

Fläche F

$$F = \int_G \sqrt{1 + (\partial_x f)^2 + (\partial_y f)^2}\, d\mu(x,y)$$

Fläche F Rotationskörper

$$F = 2\pi \int_a^b f(z) \sqrt{1 + f'(z)^2}\, dz$$

Beschreibung in Zylinderkoordinaten.

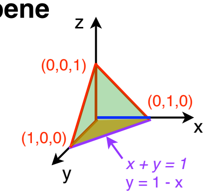
Fläche F allgemein

$$F = \int_A \left| \begin{pmatrix} \partial_x f_1 \\ \partial_x f_2 \\ \partial_x f_3 \end{pmatrix} \times \begin{pmatrix} \partial_y f_1 \\ \partial_y f_2 \\ \partial_y f_3 \end{pmatrix} \right| d\mu = \int_A \left| D_1 f \times D_2 f \right| dx\, dy$$

Volumenintegrale auf x-, y-Ebene

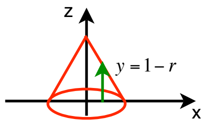
Pyramide

$$V = \int_0^1 \int_0^{1-x} (1 - x - y)\, dy\, dx$$



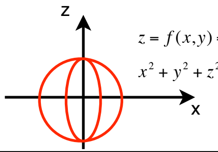
Kegel

$$V = \int_0^{2\pi} \int_0^1 (1-r)\cdot r\, dr\, d\varphi$$



Kugel

$$V = \int_0^{2\pi} \int_0^1 \sqrt{1-r^2}\cdot r\, dr\, d\varphi$$

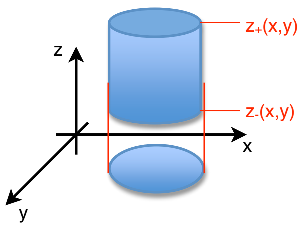


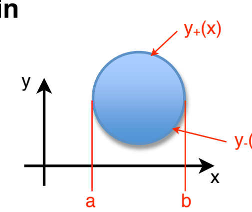
$$z = f(x,y) = \sqrt{1-x^2-y^2} = \sqrt{1-r_2}$$

$$x^2 + y^2 + z^2 = 1 \Rightarrow f(x,y) = \sqrt{1-x^2-y^2}$$

$$r^2 = x^2 + y^2$$

Volumenintegrale allgemein





$$\left\{ (x,y,z) \,\middle|\; y_-(x) \le y \le y_+(x) \wedge z_-(x,y) \le z \le z_+(x,y) \right\}$$

$$\int_V f(x,y,z)\, d\mu(x,y,z) = \int_a^b \int_{y_-(x)}^{y_+(x)} \int_{z_+(x,y)}^{z_+(x,y)} f(x,y,z)\, dz\, dy\, dz$$

Koordinatentransformation

2-dim „Flächenelement"

$$r\, dr\, d\varphi \rightarrow \int_{\dots}^{\dots} \int_{\dots}^{\dots} f(r,\varphi)\, r\, dr\, d\varphi$$

Volumenform

kartesische Koordinaten

$$dx\, dy\, dz \rightarrow \int_{\dots}^{\dots} \int_{\dots}^{\dots} \int_{\dots}^{\dots} f(x,y,z)\; dx\, dy\, dz$$

Polarkoordinaten

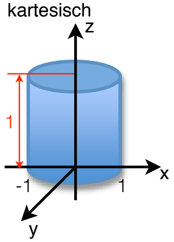
$$r\, dr\, d\varphi\, dz \rightarrow \int_{\dots}^{\dots} \int_{\dots}^{\dots} \int_{\dots}^{\dots} f(r,\varphi,z)\, r\, dr\, d\varphi\, dz$$

Kugelkoordinaten

$$r^2\cdot\sin(\vartheta)\, dr\, d\varphi\, d\vartheta \rightarrow \int_{\dots}^{\dots} \int_{\dots}^{\dots} \int_{\dots}^{\dots} f(r,\varphi,\vartheta)\, r^2\cdot\sin(\vartheta)\, dr\, d\varphi\, d\vartheta$$

Integrationsgrenzen

kartesisch

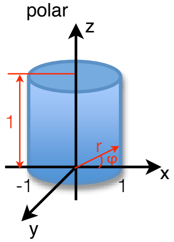


$$\underbrace{G}_{\text{Grundriss}} = \left\{ (x,y) \,\middle|\; \underbrace{-1 \le x \le 1,}_{\text{x-Wertebereich}} \; \underbrace{-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}}_{\text{y-Wertebereich abhängig von x}} \right\}$$

$$f(x,y) = 1$$

$$V = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 1\, dy\, dx$$

polar



$$\underbrace{G}_{\text{Grundriss}} = \left\{ (r,\varphi) \,\middle|\; \underbrace{0 \le \varphi \le 2\pi}_{\text{Winkel } \varphi} \wedge \underbrace{0 \le r \le 1}_{\text{Länge r}} \right\}$$

$$f(x,y) = 1$$

$$V = \int_0^{2\pi} \int_0^1 1\cdot r\, dy\, dx$$

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Schwerpunkt

$$x_S = \frac{\int_G x \cdot \rho(x,y,z)\, dx\, dy\, dz}{m}$$

$$y_S = \frac{\int_G y \cdot \rho(x,y,z)\, dx\, dy\, dz}{m}$$

$$z_S = \frac{\int_G z \cdot \rho(x,y,z)\, dx\, dy\, dz}{m}$$

Bsp.: Halbkugel
Die Halbkugel ist definiert durch [0,r]x[0,2π]x[0,π/2].
In x- und y-Richtung sind sie Rotationssymetrisch. Also sind diese Schwerpunktskoordinaten 0.
Bleibt die z-Koordinate zu berechnen:

$$z_S = \frac{1}{m}\int_H z \cdot \rho\, dx\, dy\, dz$$

$$z_S = \frac{1}{m}\int_0^r \int_0^{2\pi} \int_0^{\pi/2} r \cdot \cos(\vartheta) \cdot \rho \cdot r^2 \cdot \sin(\vartheta)\, d\vartheta\, d\varphi\, dr$$

$$z_S = \frac{\rho}{m}\int_0^r r^3\, dr \int_0^{2\pi} d\varphi \int_0^{\pi/2} \cos(\vartheta) \cdot \sin(\vartheta)\, d\vartheta$$

$$z_S = \frac{\rho}{m}\frac{r^4}{4}\pi\left[-\frac{1}{2}\cos(2\vartheta)\right]_0^{\pi/2} = \frac{\rho}{m}\frac{r^4}{4}\pi$$

$$z_S = \frac{3\rho r^4 \pi}{4 \cdot 2\pi r^3 \rho} = \underline{\underline{\frac{3}{8}r}}$$

Trägheitsmoment

$$I = \int_G r^2 \cdot \rho(x,y,z)\, dx\, dy\, dz$$

Bsp.: Rohr
Das Rohr hat eine Länge l mit Radien r_1 und r_2 und wird in Zylinderkoordinaten durch $[r_1,r_2]$x[0,2π]x[0,l] beschrieben. Es wird mit der Dichte ρ hergestellt.

$$I_{Rohr} = \int_R \rho \cdot r^2\, d\mu$$

$$I_{Rohr} = \int_{r_1}^{r_2} \int_0^{2\pi} \int_0^{l} \rho \cdot r^2\, r\, dz\, d\varphi\, dr$$

$$I_{Rohr} = 2\pi l \int_{r_1}^{r_2} r^3 dr\, \rho$$

$$\underline{\underline{I_{Rohr} = \frac{\pi}{2} l\left(r_2^4 - r_1^4\right)\rho}}$$